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Propagation of internal gravity waves in fluids with shear flow and rotation

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In a rotating system, the vertical transport of angular momentum by internal gravity waves is independent of height, except at critical levels where the Doppler-shifted wave frequency is equal to plus or minus the Coriolis frequency. If slow rotation is ignored in studying the propagation of internal gravity waves through shear flows, the resulting solutions are in error only at levels where the Doppler-shifted and Coriolis frequencies are comparable.

1. Introduction

Recently Booker & Bretherton (1967) analysed the propagation of internal gravity waves through critical levels of a fluid in shear flow. (A critical level is the level at which fluid velocity equals wave horizontal phase velocity: the wave equation is singular at this level.) They considered a Boussinesq inviscid adiabatic fluid in a non-rotating system and concluded that when the Richardson number is greater than $\frac{1}{4}$ the waves are attenuated as they pass through the critical level. They also found that the vertical flux of horizontal momentum, constant elsewhere in the fluid (Eliassen & Palm 1961), is discontinuous at the singular level, indicating a transfer of momentum to the mean flow at that point.

When these analyses are applied to geophysical problems an important assumption must be recognized: the rotation of the fluid has been taken to have negligible influence. It is not sufficient that the wave frequency as observed at the ground or some reference frame moving with the fluid be large compared to the rotational frequency for this to be valid. In the vicinity of a critical level, the Doppler-shifted wave frequency, as observed by a fluid parcel, tends to zero and must inevitably be small compared to any rotation frequency.

In fact, the singularities of a rotating system differ from those of a non-rotating system both in number and form; there are critical levels when the Doppler-shifted frequency equals plus or minus the Coriolis frequency as well as zero. It is not *a priori* evident that a wave passing through this group of critical levels will respond as though it had encountered the single critical level of the corresponding non-rotating atmosphere.

In addition, the vertical flux of horizontal momentum is not conserved in a rotating system, and thus its use as a measure of wave intensity (Booker & Bretherton 1967) is somewhat obscured.

This study has two objectives. The first is to define a flux, analogous to the momentum flux, which *is* conserved in a rotating system. It will turn out that this may be thought of as a vertical transport of angular momentum.

The second objective is to show that the solutions to the rotating system approach asymptotically those of the non-rotating system sufficiently far on either side of the critical levels, though their behaviour in the vicinity of such levels is quite dissimilar.

Together these results support more rigorously the approximation of neglecting rotation for higher frequency internal gravity waves and define more precisely when this approximation is valid.

2. Derivation of the wave equation

We shall make use of a model which is planar, Boussinesq, inviscid, and adiabatic, and with rotation about a vertical axis. Let the fluid have a mean density ρ_0 with the vertical structure

$$\partial \ln \rho_0 / \partial z = -\beta, \tag{1}$$

and a mean velocity $u_0(z)$ in the x direction. The fluid has a mean angular velocity Ω about the vertical. We shall assume there is a sinusoidal wave of small amplitude so that wave perturbation parameters have the form

$$q(x,y,z,t) = \hat{q}(z)\rho^{i(\omega t+kx+ly)}.$$

In other words, we are dealing with a single Fourier component in time and the horizontal directions.

The perturbation equations of motion are:

$$\frac{\partial u}{\partial t}+u_0\frac{\partial u}{\partial x}+w\frac{du_0}{dz} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}+2\Omega v, \tag{2}$$

$$\frac{\partial v}{\partial t}+u_0\frac{\partial v}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}-2\Omega u, \tag{3}$$

$$\frac{\partial w}{\partial t}+u_0\frac{\partial w}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}-g\frac{\rho}{\rho_0}. \tag{4}$$

Here u, v and w are the x, y and z velocities, p is pressure, ρ density and g the gravitational acceleration. Perturbation quantities are given without subscripts, zero-order quantities with the subscript 0. The incompressibility of a Boussinesq fluid $D(\rho+\rho_0)/Dt = 0$ yields

$$\frac{\partial \rho}{\partial t}+u_0\frac{\partial \rho}{\partial x}+w\frac{\partial \rho_0}{\partial z}+v\frac{\partial \rho_0}{\partial y} = 0 \tag{5}$$

and the conservation of mass requires

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0. \tag{6}$$

The mean density is gravitationally stratified in the vertical, and, as a manifestation of the thermal wind equation, shows a variation in the y direction. For geostrophic balance,

$$\frac{\partial p_0}{\partial y} = -2\Omega u_0 \rho_0 \tag{7}$$

and for hydrostatic balance

$$\frac{\partial p_0}{\partial z} = -g\rho_0. \tag{8}$$

Taking $\partial/\partial z$ of (7) and $\partial/\partial y$ of (8), we find

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial y} = \frac{2\Omega}{g}\frac{du_0}{dz} - \frac{2\Omega\beta u_0}{g}. \tag{9}$$

The last term of (9) is negligible in the Boussinesq approximation.

Equations (1) to (9) can be combined to yield the wave equation

$$\begin{aligned}[(\omega+ku_0)^2-4\Omega^2]\frac{d^2w}{dz^2} &+ \left[\frac{8\Omega^2 k}{(\omega+ku_0)} - i4\Omega l\right]\frac{du_0}{dz}\frac{dw}{dz} \\ &+ \left\{[g\beta-(\omega+ku_0)^2](k^2+l^2) + \frac{i4\Omega lk}{(\omega+ku_0)}\left(\frac{du_0}{dz}\right)^2 \right. \\ &\left. - [(\omega+ku_0)k + i2\Omega l]\frac{d^2u_0}{dz^2}\right\} w = 0. \end{aligned} \tag{10}$$

This is a minor generalization of the wave equation derived by Eady (1949), who made the additional approximations that the shear rate is constant, and that the wave is also in hydrostatic balance. (These approximations drop the term involving the second derivative of u_0 and the term $-(\omega+ku_0)^2(k^2+l^2)w$, respectively.) Equation (10) is singular if $(\omega+ku_0) = 0$, 2Ω, or -2Ω. Other wave parameters are related to w:

$$p = \frac{i[(\omega+ku_0)k+i2\Omega l](du_0/dz)w - i[(\omega+ku_0)^2-4\Omega^2](dw/dz)}{(\omega+ku_0)(k^2+l^2)}, \tag{11}$$

$$u = \frac{il^2(du_0/dz)w + i[(\omega+ku_0)k - i2\Omega l](dw/dz)}{(\omega+ku_0)(k^2+l^2)}, \tag{12}$$

$$v = \frac{-ikl(du_0/dz)w - [2\Omega k - il(\omega+ku_0)](dw/dz)}{(\omega+ku_0)(k^2+l^2)}, \tag{13}$$

$$\rho = \frac{-i\beta w}{(\omega+ku_0)} + \frac{2\Omega kl(du_0/dz)w - i2\Omega[2\Omega k - i(\omega+ku_0)l](du_0/dz)(dw/dz)}{g(\omega+ku_0)^2(k^2+l^2)}. \tag{14}$$

If ξ and η are respectively the x and y displacements of a fluid parcel from its rest position,

$$\xi = \frac{l^2(du_0/dz)w + [(\omega+ku_0)k - i2\Omega l](dw/dz)}{(\omega+ku_0)^2(k^2+l^2)}, \tag{15}$$

$$\eta = \frac{-kl(du_0/dz)w + i[2\Omega k - il(\omega+ku_0)](dw/dz)}{(k^2+l^2)(\omega+ku_0)^2}. \tag{16}$$

3. Angular momentum flux

The equations of motion presume rotation about some vertical axis, but do not depend on the specific location of that axis. Let the mean position of a fluid parcel be at X_0, Y_0, Z_0, and its instantaneous horizontal displacement from this position be ξ, η. Let the axis of rotation be located at the origin. Then the instantaneous angular momentum per unit volume of the fluid parcel is

$$[(X_0+\xi)^2+(Y_0+\eta)^2]\,\Omega+(X_0+\xi)\,v-(Y_0+\eta)\,(u_0+u)$$
$$\simeq (X_0^2+Y_0)^2\,\Omega-Y_0u_0+X_0(v+2\Omega\xi)-Y_0(u-2\Omega\eta), \quad (17)$$

to first order and making the approximation that $u_0 \ll 2\Omega Y_0$. If (17) is multiplied by $\rho_0 w$ and a time average is taken, the resultant mean vertical flux of angular momentum is

$$\rho_0 X_0\overline{(v+2\Omega\xi)\,w}-\rho_0 Y_0\overline{(u-2\Omega\eta)\,w}$$
$$= \tfrac{1}{2}\rho_0 X_0 \operatorname{Re} w^*(v+2\Omega\xi)-\tfrac{1}{2}\rho_0 Y_0 \operatorname{Re} w^*(u-2\Omega\eta), \quad (18)$$

where an overbar denotes a time average for the parcel and w^* is the complex conjugate of w. By substitution of (12), (13), (15) and (16), we find

$$\operatorname{Re} w^*(v+2\Omega\xi) = \frac{l}{(k^2+l^2)}\,G \quad (19)$$

and

$$\operatorname{Re} w^*(u-2\Omega\eta) = \frac{k}{(l^2+k^2)}\,G, \quad (20)$$

where

$$G = \operatorname{Re}\left\{\frac{2\Omega l}{(\omega+ku_0)^2}\frac{du_0}{dz}ww^*+\frac{i[(\omega+ku_0)^2-4\Omega^2]}{(\omega+ku_0)^2}w^*\frac{dw}{dz}\right\}. \quad (21)$$

If one differentiates (21) with respect to z, and substitutes (10) into the result, then, *providing* k, l, ω and Ω are all real, the result is

$$dG/dz = 0. \quad (22)$$

Thus $\operatorname{Re} w^*(v+2\Omega\xi)$, $\operatorname{Re} w^*(u-2\Omega\eta)$ and the vertical flux of angular momentum are constant with height for a real frequency and horizontal wavenumber. This is true at any level except at singular levels, where substitution of (10) is invalid.

We also note that

$$\lim_{|(\omega+ku_0/2\Omega|\to\infty} \operatorname{Re} w^*(v+2\Omega\xi) \to \operatorname{Re} w^*v \quad (23)$$

and

$$\lim_{|(\omega+ku_0)/2\Omega|\to\infty} \operatorname{Re} w^*(u-2\Omega\eta) \to \operatorname{Re} w^*u, \quad (24)$$

so that, far away from the singular levels, the vertical transports of angular and linear momentum are almost equivalent, and the latter is almost constant.

4. Asymptotic solutions to the wave equation

With the aid of two further assumptions, it is a straightforward process to obtain asymptotic solutions to (10) in regions well away from its singular points. We shall assume that: (i) the velocity shear, du_0/dz, is constant with height, (ii) the Doppler-shifted frequency $(\omega+ku_0)$ is of small magnitude compared to the Brunt frequency, $(g\beta)^{\frac{1}{2}}$.

Without any loss of generality, we can take $z = 0$ at the height at which $(\omega+ku_0) = 0$ and write (10) as

$$\left[\left(k\frac{du_0}{dz}\right)^2 z^2-4\Omega^2\right]\frac{d^2w}{dz^2}+\left[\frac{8\Omega^2}{z}-i4\Omega l\frac{du_0}{dz}\right]\frac{dw}{dz}+\left[g\beta(k^2+l^2)+\frac{i4\Omega l}{z}\frac{du_0}{dz}\right]w = 0. \quad (25)$$

Power series solutions for (25) in descending powers of z may be obtained; $\zeta = 1/z$ is substituted and the resulting equation can be solved in a power series expansion about $\zeta = 0$ by the method of Frobenius. The expansion is valid in the range

$$\left|\frac{k}{2\Omega}\frac{du_0}{dz}\frac{1}{\zeta}\right| = \left|\frac{k}{2\Omega}\frac{du_0}{dz}z\right| > 1. \tag{26}$$

The resulting expansion, in terms of z, is

$$w = a_0 z^a[1+a_1 z^{-1}+a_2 z^{-2}+\ldots]+b_0 z^b[1+b_1 z^{-1}+b_2 z^{-2}+\ldots], \tag{27}$$

where a_0 and b_0 are arbitrary integration constants,

$$\begin{aligned} a &= \tfrac{1}{2}+i\mu, \quad b = \tfrac{1}{2}-i\mu, \quad \mu = (J'-\tfrac{1}{4})^{\frac{1}{2}}, \\ J' &= g\beta\left(1+\frac{l^2}{k^2}\right)\Big/\left(\frac{du_0}{dz}\right)^2, \quad a_1 = b_1 = -\frac{i\Omega l}{k^2(du_0/dz)}, \\ a_2 &= \frac{a}{2}\left(\frac{a+3}{2a+3}\right)-\frac{l^2}{k^2}\left(\frac{a+2}{2a+3}\right), \quad b_2 = \frac{b}{2}\left(\frac{b+3}{2b+3}\right)-\frac{l^2}{k^2}\left(\frac{b+2}{2b+3}\right). \end{aligned} \tag{28}$$

J' is a modified Richardson number. The coefficients a and b are in general of magnitude unity or a little larger. Therefore, if z is such that

$$|\omega+ku_0| = \left|k\frac{du_0}{dz}z\right| \gg \left|2\Omega\left(1+\frac{l^2}{k^2}\right)^{\frac{1}{2}}\right|, \tag{29}$$

equation (27) is approximately

$$w \simeq c_1 z^{\frac{1}{2}+i\mu}+c_2 z^{\frac{1}{2}-i\mu}. \tag{30}$$

In the absence of rotation, (25) becomes

$$z^2\frac{d^2w}{dz^2}+J'w = 0, \tag{31}$$

with the solutions
$$w = c_3 z^{\frac{1}{2}+i\mu}+c_4 z^{\frac{1}{2}-i\mu}, \tag{32}$$

where c_3 and c_4 are again integration constants.

Let us prescribe boundary conditions at point A, say, of figure 1, so that the integration constants a_0 and b_0 of (27) are known. If we can follow a path, such as the dotted line, through the (shaded) region of validity of (27) in the complex z-plane we can use this equation to determine w at A'.

To the extent that (30) is a valid approximation for (27) along this path, that is to the extent that inequality (29) holds, the solutions at A with and without rotation will be identical. This follows from the similarity of (30) and (31). Inequality (29) can be met increasingly well as A and A' are moved away from the singular points.

In integrating (25) or (31) through singularities, it is necessary to determine whether to pass above or below the singularity in the complex plane. Booker & Bretherton present causality arguments in favour of the addition of a small negative imaginary component to ω; as a result the integration is carried below the singularities. (The same result is also justifiable on the basis of Rayleigh damping. See appendix.)

The same argument can be applied here. Thus the path of integration is shown by the solid curve A–A' of figure 1. As the integration path lies below the singularities,

$$z \simeq |z| e^{-i\pi} \quad (z < 0). \tag{33}$$

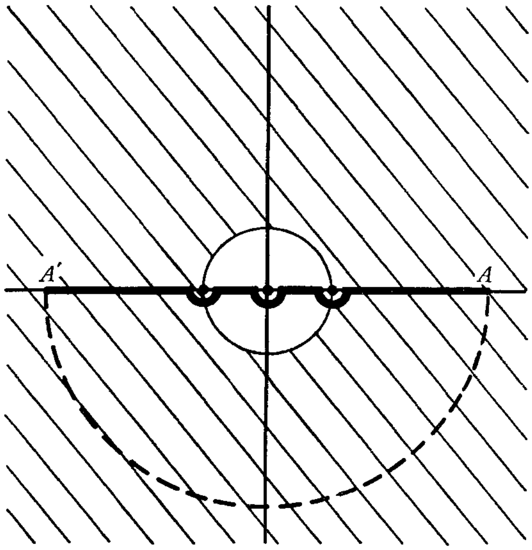

FIGURE 1. Paths of integration in the complex z-plane around the singularities (●) of (25). Shaded area is region of validity of asymptotic solution given by (27).

It is also possible to obtain power series expansions of (10) about its singularities at $\omega + ku_0 = 0,\ 2\Omega,\ -2\Omega$. If ξ_1 is the vertical distance from the height at which $\omega + ku_0 = 0$, then in this neighbourhood

$$w = c_5[1 + O(\xi_1)] + c_6 \xi_1^3 [1 + O(\xi_1)]. \tag{34}$$

If this is inserted into (21), one finds that the leading term in a power series for G is a constant, continuous across the discontinuity. (Higher powers of ξ_1 must of course cancel for real k, l and ω.) In this case G and the angular momentum transport are continuous across the discontinuity.

If instead ξ_2 is the distance to the level at which $\omega + ku_0 = 2\Omega$, the series solutions in this neighbourhood are

$$w = c_7[1 + O(\xi_2)] + c_8 \xi_2^{il/k} [1 + O(\xi_2)], \quad (l \neq 0); \tag{35}$$

or

$$w = c_9[1 + O(\xi_2)] + c_{10} \ln \xi_2 [1 + O(\xi_2)], \quad (l = 0). \tag{36}$$

Substitution of these expressions in (21) shows that in general G will be discontinuous. A similar result holds for $\omega + ku_0 \rightarrow -2\Omega$.

5. A numerical illustration

Equation (10) was solved numerically by Mr Larry Williams for a specific case in order to illustrate the asymptotic behaviour of waves. The problem was nondimensionalized by taking $g = 1$, $\beta = 1$. The mean velocity shear was as shown in figure 2. In this example $\omega = -i \times 10^{-4}$, $2\Omega = 10^{-2}$, $k = 1$, $l = 0$, $u_0 = 0{\cdot}1$, $L = 10^{-1}\pi$. Thus the shear at $z = 0$ corresponded to a Richardson number of one.

It was assumed that there was an unspecified source for an upward-propagating wave (with $\overline{wp}$ positive) located somewhere remotely in the lower half-space, and that there were also a transmitted upward propagation wave in the upper half-

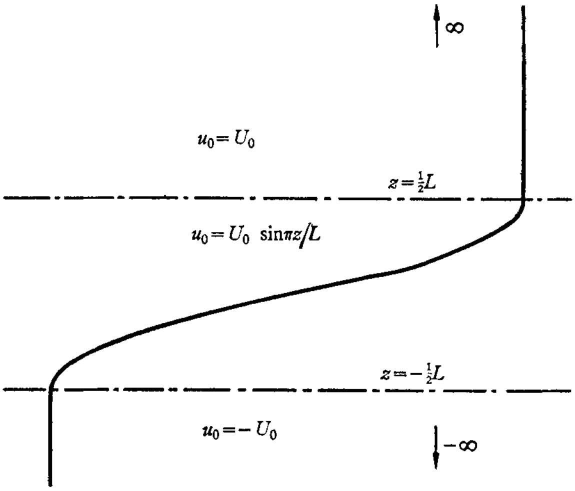

FIGURE 2. Mean velocity profile as a function of height for numerical experiment.

space and a reflected wave in the lower half-space. The transmitted wave was arbitrarily set equal to unity at $z = \frac{1}{2}L$, thus giving w and dw/dz as two boundary conditions. $w(z)$ was computed by using Hamming's (1959) method, using a grid interval $\Delta z = 10^{-5}$ and the compatible incident and reflected waves obtained in the lower half-space. Tests showed that the results were not sensitive either to the magnitudes of Δz or the imaginary part of ω. Incident and reflected wave amplitudes were obtained for the lower half-space.

Figures 3*a* and 3*b* show the behaviour of $w(z)$ through the shear zone in this case and in a similar case where $2\Omega = 0$. Although the waves behave quite differently near $z = 0$, they are quite similar on *either* side. The ratio of linear momentum flux in the upper and lower half-spaces is 1:238·8 in the rotating case, and 1:237·5 in the non-rotating case. (The ratio of angular momentum fluxes in the rotating case is the same, 1:238·8, since $(\omega - ku_0)^2$ is the same in both half-spaces.)

These figures are quite comparable with Booker & Bretherton's estimates for a non-rotating system. These authors find that the momentum flux of an upward-going wave is attenuated by a factor $e^{-2\pi\mu}$ or 1:230·8. One referee has kindly pointed out that there will be partial reflexions near the 'knees' of the mean wind profile at $z = \pm\frac{1}{2}L$ of figure 2. Let A be the momentum flux of the upward wave mode above $z = 0$ but below the upper knee, and B be the flux of the downward wave in this region. B results from reflexion at the upper knee. Then the net flux just above the critical layer is $A - B$ and $-A\,e^{2\pi\mu} + B\,e^{-2\pi\mu}$ will be the flux just below the critical layer. The magnitude of the momentum flux ratios will then be

$$\sim \; + e^{-2\pi\mu}\left(1 + \frac{B}{A}\right).$$

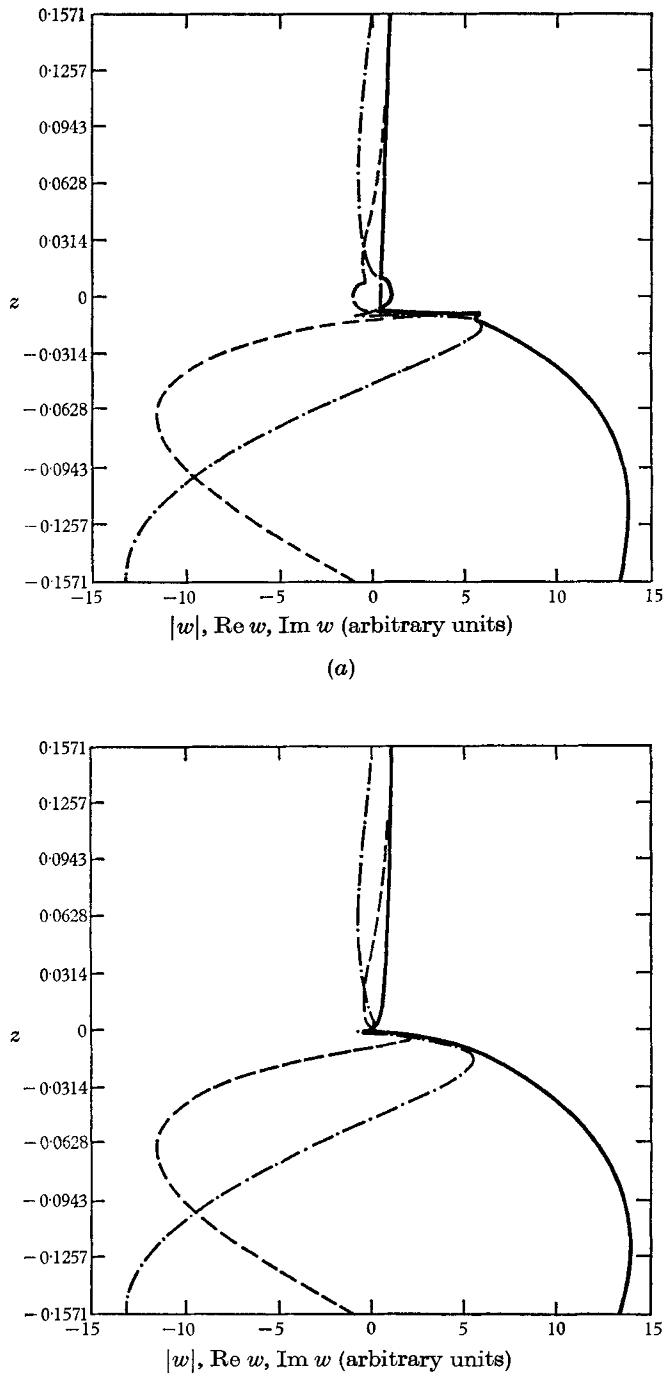

FIGURE 3. Numerical solutions for w as a function of height z for (*a*) rotating, and (*b*) non-rotating model atmospheres. Vertical distances are in units of the density scale height. ——, $|w|$; – – – –, Re w; –·–·–, Im w.

The ratio of downgoing to upgoing momentum fluxes in the lower half-space is 2·64 % for the non-rotating and 2·66 % for the rotating case. This is a good estimate of the reflexion coefficient at the lower knee and the best estimate available for B/A. We find

$$1:e^{-2\pi\mu}(1+0\cdot 0264) \simeq 1:236\cdot 9.$$

Thus the computed flux ratios differ from each other and from a value determined from the Booker–Bretherton theory combined with numerical reflexion coefficients by factors of the order of 1 %. One may still be critical and ask what the source of these small differences may be. Numerical experiments varying Δz and also integration using a simpler finite difference scheme change results only by small fractions of a per cent. Hence it does not appear that the numerical computations *per se* can account for the differences. However, changes in the imaginary part of ω produced changes of the order of 1 % in the numerical results.

It thus appears to us that the discrepancy between momentum flux ratios for the numerical and Booker–Bretherton analyses of the non-rotating case arises in part through the use of a finite imaginary component of frequency and also in part to poor estimates of the partial reflexions of the wave. The further difference of the rotating system momentum flux ratio may stem from these sources and also from the approximations of our analysis, especially in going from (27) to (30). We have not tried to assess the relative importance of these influences.

6. Conclusions

In a rotating system, it is the vertical transport of angular momentum by internal gravity waves that is conserved. Rotation of such a system with linear velocity shear can be ignored, providing that attention is confined to regions where the Doppler-shifted wave frequency, $\omega + ku_0$, satisfies the relation

$$|\omega + ku_0| \gg 2\Omega(1+l^2/k^2)^{\frac{1}{2}}. \tag{37}$$

In zones where this condition does not hold, the rotating and non-rotating systems may show widely differing solutions which, however, converge on either side of the zone.

Mr Larry Williams developed and programmed the numerical example used in this study. Dr Akira Kasahara provided a number of helpful comments during the preparation of the manuscript.

Appendix. Rayleigh damping

Let us assume that the fluid is no longer inviscid, but has a Rayleigh viscous force directly proportional to the fluid velocity, and a mass diffusion loss directly proportional to the mass perturbation. Then (2)–(5) become

$$\frac{\partial u}{\partial t} + u_0\frac{\partial u}{\partial x} + w\frac{\partial u_0}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + 2\Omega v - \frac{1}{\tau_1}u, \tag{38}$$

$$\frac{\partial v}{\partial t} + u_0\frac{\partial v}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - 2\Omega u - \frac{1}{\tau_1}v, \tag{39}$$

$$\frac{\partial w}{\partial t}+u_0\frac{\partial w}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}-g\frac{\rho}{\rho_0}-\frac{1}{\tau_1}w, \tag{40}$$

$$\frac{\partial \rho}{\partial t}+u_0\frac{\partial \rho}{\partial x}+w\frac{\partial \rho_0}{\partial y}+v\frac{\partial \rho_0}{\partial y} = -\frac{1}{\tau_2}\rho, \tag{41}$$

where τ_1 and τ_2 are respectively the viscous and diffusional time constants. If in addition $\tau_1 = \tau_2 = 1/\omega_i$ = constant, the resulting wave equation is identical to (10) except that ω must be replaced by $\omega - i\omega_i$.

In this case, the momentum flux is no longer a constant. Solutions correspond to lines of negative ω_i on figures 1 and 2. If $|\omega_i| \ll |2\Omega|$ the influence of rotation is negligible.

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